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Tilting classes over wild hereditary algebras ☆

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Let A be a ring and T be a right A-module. Then T is a *tilting module* provided that $\operatorname{p.dim} T \leqslant 1$, $\operatorname{Ext}^1_A(T, T^{(I)}) = 0$ for any set I, and there is a short exact sequence $0 \to A \to T_0 \to T_1 \to 0$ where T_0 and T_1 are direct summands in a direct sum of (possibly infinitely many) copies of T. Equivalently, T is tilting if and only if $\operatorname{Gen}(T) = \{T\}^\perp$ [7]. Here, $\operatorname{Gen}(T)$ denotes the class of all homomorphic images of direct sums of copies of T, and, for a class of modules $\mathcal{C}$,

$$\mathcal{C}^\perp = \operatorname{Ker}\operatorname{Ext}^1_A(\mathcal{C}, -) = \left\{M \in \text{Mod-}A \mid \operatorname{Ext}^1_A(C, M) = 0 \text{ for all } C \in \mathcal{C}\right\}.$$

If T is a tilting module then $\{T\}^\perp$ is a torsion class in Mod-A, the *tilting class* generated by T. If T' is another tilting module then T is said to be *equivalent* to T' if $\{T\}^\perp = \{T'\}^\perp$.

Tilting classes are characterized as the torsion classes that are special preenveloping in Mod-A, see [3]. In particular, given any set $\mathcal{S}$ of finitely presented modules of projective dimension at most 1, the class $\mathcal{S}^\perp$ is always a tilting class; however, there need not exist any finitely presented tilting module T such that $\mathcal{S}^\perp = \{T\}^\perp$.

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This phenomenon occurs already in the setting of modules over hereditary Artin algebras: Ringel proved that if A is a tame hereditary algebra over an algebraically closed field k and $\mathcal{R}$ is the set of all regular modules, then the tilting class of all divisible modules, $\mathcal{D} = \mathcal{R}^{\perp}$, is generated by the tilting module $T_R = G \oplus \bigoplus_{\lambda} R_{\lambda}$, where G is the generic module and $\{R_{\lambda} \mid \lambda \in k \cup \{\infty\}\}$ is the set of all Prüfer modules, cf. [2, Example 1.4], [18], and [21]. Denote by n the endolength of G. Then for each set of tubes, S, there is a tilting module $T_S = G_S \oplus P_S$ such that $S^{\perp} = \{T_S\}^{\perp}$ where P_S is the direct sum of the Prüfer modules corresponding to the tubes in S, and G_S is defined by $A \subseteq G_S \subseteq G^n$ and $G_S/A \cong P_S$. Notice that T_S is equivalent to T_R in the case when S is the set of all tubes; however, if $S \neq \emptyset$ then T_S is not equivalent to any finitely generated tilting module, and $T_{S'}$ is not equivalent to T_S for $S' \neq S$.

In this paper, we investigate tilting classes over connected hereditary algebras of infinite representation type, and in particular, over connected wild hereditary algebras. The case when the generating tilting module is finitely generated was studied in [13]. In [17], Lukas proved several facts important for our general setting. In the present terminology, he showed that given a wild hereditary algebra A, the classes of all divisible modules, and all $\mathcal{P}^{\infty}$-torsion modules, are tilting classes.

Though several results will be proved in a more general setting, we will mainly consider the tilting classes of the form $\mathcal{S}^{\perp}$ for a set of finitely presented modules, $\mathcal{S}$, over a hereditary Artin algebra A. Without loss of generality, $\mathcal{S} \subseteq$ ind-A where ind-A denotes a representative set of all non-zero finitely generated indecomposable modules.

We will primarily be interested in the question of when $\mathcal{S}^{\perp} = \{F\}^{\perp}$ for a finitely generated tilting module F, and in case there is no such F, in an explicit construction of an infinitely generated tilting module T with $\mathcal{S}^{\perp} = \{T\}^{\perp}$. We will have a complete answer in case $\mathcal{S}$ consists of preprojective or preinjective modules, and give partial answers in case $\mathcal{S}$ consists of regular modules.

1. Hereditary Artin algebras and torsion pairs

For a commutative artinian ring k, a k-algebra A is called an *Artin algebra*, if it is finitely generated as k-module. Additionally we will assume that A is a faithful k-module and that A is connected. This means that 0 and 1 are the only central idempotents in A, in particular, k is a local ring.

By Mod-A, we denote the category of all (right A-) modules, and by mod-A the subcategory of all finitely presented modules. Also, $\tau = \mathrm{D}\,\mathrm{Tr}$ and $\tau^{-} = \mathrm{Tr}\,\mathrm{D}$, denote the Auslander–Reiten translations in mod-A. By Auslander–Reiten formula, we get an epimorphism $\mathrm{Hom}_A(Y, \tau X) \to \mathrm{D}\,\mathrm{Ext}^1_A(X, Y)$ which is an isomorphism if X has projective dimension at most 1. Similarly, the epimorphism $\mathrm{Hom}_A(\tau^{-} Y, X) \to \mathrm{D}\,\mathrm{Ext}^1_A(X, Y)$ is an isomorphism if Y has injective dimension at most 1.

The *Auslander–Reiten quiver*, $\Gamma(A)$, is a directed graph whose set of vertices is ind-A, and whose arrows are induced by the Auslander–Reiten sequences $0 \to \tau X \to E \to X \to 0$ for $X \in$ ind-A non-projective, and by the embeddings $\mathrm{rad}\, X \subseteq X$ for $X \in$ ind-A projective. For more details, see [1].

Moreover, if A is hereditary then k is a field, and $\tau^{-} = \mathrm{Ext}_A^1(\mathrm{D}(A), -)$ and $\tau = \mathrm{D}\,\mathrm{Ext}_A^1(-, A) \cong \mathrm{Tor}_1^A(\mathrm{D}(A), -)$ are endo-functors on mod-A. The Auslander–Reiten formula can then be extended as follows, see [8,17].

Lemma 1.1. *If A is a hereditary Artin algebra then* $\mathrm{D}\,\mathrm{Ext}_A^1(X, M) \cong \mathrm{Hom}_A(M, \tau X)$ *and* $\mathrm{Ext}_A^1(M, X) \cong \mathrm{D}\,\mathrm{Hom}_A(\tau^{-} X, M)$ *for $X \in$ mod-A and $M \in$ Mod-A.*

Assume A is hereditary and representation-infinite. Then $\Gamma(A)$ is partitioned into three types of modules: a module $X \in$ ind-A is *preprojective* (*preinjective*) if $\tau^m X = 0$ ($\tau^{-m} X = 0$) for some $m \geqslant 0$; X is *regular* if $\tau^m \tau^{-m} X \cong X$ for all integers m. A module $M \in$ mod-R is *preprojective* (*preinjective*, and *regular*) if either $M = 0$, or each indecomposable direct summand of M is isomorphic to a preprojective (preinjective, and regular) module in ind-A. The set of all $M \in$ mod-R that are preprojective (preinjective, and regular) will be denoted by $\mathcal{P}$ ($\mathcal{I}$, and $\mathcal{R}$).

The Auslander–Reiten quiver $\Gamma(A)$ consists of infinitely many (connected) components: one preprojective component, whose vertices are the indecomposable preprojective modules, one preinjective component, with $\mathcal{I} \cap$ ind-A as vertices, and an infinite set of regular components (with vertices $\mathcal{R} \cap$ ind-A).

If A is tame hereditary, all regular components are tubes, all of them homogeneous, up to finitely many. If A is wild hereditary, all regular components are of type ZA_∞. In both cases, the modules at the border of the regular components are called *quasi-simple*. If Y is an arbitrary module contained in a regular component $\mathcal{C}$, there exists a unique quasi-simple module X in $\mathcal{C}$ and a chain of irreducible monomorphisms

$$X = X(1) \to X(2) \to \cdots \to X(r) = Y, \tag{$*$}$$

which we will consider as inclusions. The number r is called the *quasi-length* of Y, and $X(i)/X(i-1) \cong \tau^{-i+1} X$ holds for $1 < i \leqslant r$.

If A is tame hereditary then $\mathcal{R}$ is a serial abelian length-category, and the quasi-simple modules form a representative set of its simple objects. If Y is an arbitrary indecomposable regular module, then there is a chain of irreducible monomorphisms as in $(*)$ forming a composition series of Y. The tubes are pairwise orthogonal, that is, $\mathrm{Hom}_A(\mathcal{T}_1, \mathcal{T}_2) = \mathrm{Ext}_A^1(\mathcal{T}_1, \mathcal{T}_2) = 0$ where $\mathcal{T}_1$ and $\mathcal{T}_2$ are different tubes.

If A is wild hereditary, the category $\mathcal{R}$ is closed under extensions and homomorphic images, but not closed under kernels and cokernels. Thus it is not abelian.

We collect further results on representation-infinite hereditary Artin algebras, used in the paper. For proofs, see, for example, [1,14–17,22].

A module X is called a *brick* if $\mathrm{End}_A(X)$ is a division ring, and it is called *sincere* provided that all simple modules occur as composition factors of X, or equivalently, if $\mathrm{Hom}_A(P, X) \neq 0$ for all indecomposable projective modules P.

Proposition 1.2. (A) *Let A be a representation-infinite hereditary Artin algebra.*

(1) *Each component of the Auslander–Reiten quiver $\Gamma(A)$ contains at most finitely many non-sincere modules.*

(2) *If Y is preinjective and* $\mathrm{Hom}_A(Y, M) \neq 0$*, then M has a non-zero preinjective direct summand. If X is preprojective and* $\mathrm{Hom}_A(M, X) \neq 0$*, then M has a non-zero preprojective direct summand.*
(3) *There exists a regular tilting module T if and only if A is wild hereditary with at least three simple modules.*
(4) *The Auslander–Reiten translation τ induces an equivalence on $\mathcal{R}$.*

(B) *Let A be wild hereditary and X, Y be non-zero regular modules. Then the following holds.*

(1) $\mathrm{Hom}_A(\tau^m X, Y) = 0$ *for* $m \gg 0$.
(2) $\mathrm{Hom}_A(X, \tau^m Y)$ *contains a monomorphism for* $m \gg 0$.
(3) *There exists a natural number $t = t(A)$ such that* $\mathrm{Hom}_A(X, Y) \neq 0$ *implies* $\mathrm{Hom}_A(X, \tau^m Y) \neq 0$ *for all* $m \geqslant t$.
(4) *If X is a quasi-simple brick, then* $\mathrm{Hom}_A(X, \tau^{-i} X) = 0$ *for all* $i > 0$.

In an abelian category $\mathcal{A}$, we call a pair $(\mathcal{T}, \mathcal{F})$ of classes of objects in $\mathcal{A}$ a *torsion pair* if $\mathrm{Hom}(\mathcal{T}, \mathcal{F}) = 0$, and both classes are maximal with respect to this property which means that for any object $M \in \mathcal{A}$, there is a short exact sequence

$$0 \to t(M) \to M \to f(M) \to 0,$$

with $t(M) \in \mathcal{T}$ and $f(M) \in \mathcal{F}$. An object $P \in \mathcal{T}$ is called Ext-*projective* (in $\mathcal{T}$), provided $\mathrm{Ext}^1_{\mathcal{A}}(P, \mathcal{T}) = 0$, that is, $\mathcal{T} \subseteq \{P\}^\perp$.

If T is a tilting module over a hereditary Artin algebra A, then the objects in add T are Ext-projective in $\mathrm{Gen}(T)$. So if $\mathrm{Gen}(T) \cap \text{mod-}A$ contains no non-zero Ext-projective modules then T has no finitely generated indecomposable direct summands.

For an Artin algebra A, we are mainly interested in tilting classes in Mod-A of the form $\mathcal{S}^\perp$ where $\mathcal{S} \subseteq \text{mod-}A$ consists of modules of projective dimension at most one.[1] We will frequently use the following well-known and easy facts:

Lemma 1.3. *Let A be a ring.*

(1) *If* $0 \to U \to M \to V \to 0$ *is a short exact sequence in* Mod-A *then* $\{U\}^\perp \cap \{V\}^\perp \subseteq \{M\}^\perp$*. If A is right hereditary then* $\{M\}^\perp \subseteq \{U\}^\perp$.
(2) *If M has a smooth filtration of length κ* $0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M = \bigcup_{i<\kappa} M_i$ *for some cardinal κ, then* $\bigcap_{i<\kappa} \{M_{i+1}/M_i \mid i < \kappa\}^\perp \subseteq \{M\}^\perp$.
(3) *If $\mathcal{S}$ is a set of finitely presented modules of projective dimension at most* 1 *then $\mathcal{S}^\perp$ is a torsion class.*

[1] Added in proof: Bazzoni and Herbera have recently announced that any tilting class over any ring A is of this form. So, for example, Theorem 2.1 below characterizes all tilting classes over Artin algebras.

2. Constructions of tilting modules

In this section, A denotes an Artin algebra over a commutative artinian ring k. We start by observing that the tilting classes under consideration correspond 1–1 to the torsion classes in mod-A containing all finitely generated injective modules:

Theorem 2.1. *Let A be an Artin algebra. There is a bijective correspondence between*

(1) *tilting torsion classes $\mathcal{C} \subseteq \text{Mod-}A$ of the form $\mathcal{C} = \mathcal{S}^{\perp}$ where $\mathcal{S}$ is a set of finitely generated modules of projective dimension $\leqslant 1$, and*
(2) *torsion classes $\mathcal{T} \subseteq \text{mod-}A$ such that $\mathcal{T}$ contains all finitely generated injective modules.*

The correspondence is given by the mutually inverse maps $\alpha : \mathcal{C} \mapsto \mathcal{C} \cap \text{mod-}A$ and $\beta : \mathcal{T} \mapsto \operatorname{Ker}\operatorname{Hom}_A(-, \mathcal{F})$ where $(\mathcal{T}, \mathcal{F})$ is a torsion pair in mod-A.

Proof. Clearly, α is well-defined.

Let $\mathcal{T}$ be as in (2) with the corresponding torsion pair $(\mathcal{T}, \mathcal{F})$ in mod-A. Then $\mathcal{T}$ contains all finitely generated cosyzygies of all simple modules, hence ${}^{\perp}\mathcal{T}$ consists of modules of projective dimension $\leqslant 1$. Indeed, if S is a simple module with injective hull $E(S)$, consider the short exact sequence $0 \to S \to E(S) \to Q \to 0$. For $M \in {}^{\perp}\mathcal{T}$ one has $0 = \operatorname{Ext}^1_A(M, Q) \cong \operatorname{Ext}^2_A(M, S)$. Since $\operatorname{Ext}^2_A(M, S) = 0$ holds for all simple modules S, we get $\operatorname{p.dim} M \leqslant 1$ by [23, Proposition 1.4]. The Auslander–Reiten formula then gives, for each $M \in \text{mod-}A$, the equivalence $M \in {}^{\perp}\mathcal{T}$ iff $\operatorname{Ext}^1_A(M, \mathcal{T}) = 0$ iff $\operatorname{Hom}_A(\mathcal{T}, \tau M) = 0$ iff $\tau M \in \mathcal{F}$. Put $\tau^{-}\mathcal{F} = \{M \in \text{mod-}A \mid \tau M \in \mathcal{F}\}$. Since $\mathcal{F}$ contains no non-zero injective modules, we have $\tau(\tau^{-} F) = F$ for each $F \in \mathcal{F}$. As $\tau^{-}\mathcal{F}$ consists of modules of projective dimension $\leqslant 1$, the Auslander–Reiten formula yields $\beta(\mathcal{T}) = \operatorname{Ker}\operatorname{Hom}_A(-, \tau(\tau^{-}\mathcal{F})) = (\tau^{-}\mathcal{F})^{\perp}$, and β is well-defined.

Clearly, $\mathcal{T} = \{M \in \text{mod-}A \mid \operatorname{Hom}_A(M, F) = 0 \text{ for all } F \in \mathcal{F}\} = \alpha\beta(\mathcal{T})$.

Conversely, let $\mathcal{C}$ be as in (1). Let $\mathcal{T} = \alpha(\mathcal{C})$, $(\mathcal{T}, \mathcal{F})$ be a torsion pair in mod-A, and $\mathcal{D} = \beta\alpha(\mathcal{C})$. Then $\alpha(\mathcal{D}) = \alpha\beta\alpha(\mathcal{C}) = \alpha(\mathcal{C})$, that is, the finitely generated modules in $\mathcal{C}$ and $\mathcal{D}$ coincide.

We claim that also the pure-injective modules in $\mathcal{C}$ and $\mathcal{D}$ coincide. To see this, let M be a module and $(f_i : M \to F_i \mid i \in I)$ a representative set (up to isomorphism) of all epimorphisms from M onto a finitely generated module. Then any homomorphism from M to a finitely generated module can be factorized through $f : M \to \prod_{i \in I} F_i$, hence f is a pure embedding (cf. [8, 2.2(c)]). Since $\mathcal{C}$ is a torsion class in Mod-A, we infer that a pure-injective module M belongs to $\mathcal{C}$ iff M is a direct summand in a (possibly infinite) direct product of elements of $\alpha(\mathcal{C})$, and similarly for $\mathcal{D}$. However, $\alpha(\mathcal{C}) = \alpha(\mathcal{D})$, so the claim follows.

Since $\mathcal{D} = (\tau^{-}\mathcal{F})^{\perp}$, the classes $\mathcal{C} = \mathcal{S}^{\perp}$ and $\mathcal{D}$ are closed under pure submodules, direct products and direct limits, so they are definable subcategories of Mod-A in the sense of [8, 2.3]. In particular, a module belongs to $\mathcal{C}$ if and only if its pure-injective envelope does, and similarly for $\mathcal{D}$. It follows that $\mathcal{C} = \mathcal{D}$, that is, $\mathcal{C} = \beta\alpha(\mathcal{C})$. $\square$

It is well known that torsion classes in Mod-A form a complete lattice. By Theorem 2.1, this also holds for the tilting torsion classes $\mathcal{C}$ as in (1).

A classical result of Assem [4] says that the tilting torsion classes in mod-A (that is, the classes $\mathcal{T} \subseteq$ mod-A of the form $\mathcal{T} = \{T\}^{\perp} \cap$ mod-A for a finitely generated tilting module T) coincide with the classes $\mathcal{T}$ as in (2) which are moreover generated by a single finitely generated module.

In other words, given $\mathcal{S} \subseteq$ ind-A, $\mathcal{S}^{\perp} = \{T\}^{\perp}$ for a finitely generated tilting module T iff $\mathcal{S}^{\perp} \cap$ mod-A is generated by a single module from mod-A. The latter condition does not hold in general, so naturally, a question arises of constructing an (infinitely generated) tilting module T with $\mathcal{S}^{\perp} = \{T\}^{\perp}$.

General results of approximation theory of infinitely generated modules provide a construction of this kind, cf. [3,9]. We will now show that a modification of this general construction in the Artin algebra case always yields a $\leqslant \kappa$-generated tilting module T in the case when $\mathcal{S}$ has cardinality $\leqslant \kappa$ where κ is an infinite cardinal.

Theorem 2.2. *Let A be an Artin algebra and κ be an infinite cardinal. Let $\mathcal{S}$ be a subset of cardinality $\leqslant \kappa$ in* ind-A *consisting of modules of projective dimension at most* 1. *Then there is a $\leqslant \kappa$-generated tilting module T such that $\mathcal{S}^{\perp} = \{T\}^{\perp}$.*

Moreover, T is a union of $< \kappa$-generated submodules of a smooth chain $(T_\alpha \mid \alpha < \kappa)$ such that $T_0 = A$, and, for each $\alpha < \kappa$, $T_{\alpha+1}/T_\alpha$ is isomorphic to a direct sum of $< \kappa$ copies of a single module $S_\alpha \in \mathcal{S}$.

Proof. Let $(S_\alpha \mid \alpha < \kappa)$ be a list of elements of $\mathcal{S}$ such that each element of $\mathcal{S}$ is listed κ times. For each $\alpha < \kappa$, let $\mathcal{E}_\alpha : 0 \to K_\alpha \subseteq F_\alpha \to S_\alpha \to 0$ be a short exact sequence such that F_α is free and finitely generated (and hence K_α is finitely generated and projective), and $\mathcal{E}_\alpha = \mathcal{E}_\beta$ provided that $S_\alpha = S_\beta$.

By induction, we define a smooth chain of $< \kappa$-generated modules $(P_\alpha \mid \alpha < \kappa)$ as follows: $P_0 = A$. Given P_α, let G_α be a generating set of the k-module $\mathrm{Hom}_A(K_\alpha, P_\alpha)$. W.l.o.g., G_α has cardinality $< \kappa$ (and $G_\alpha \neq \emptyset$ since K_α is projective and $A \subseteq P_\alpha$). Denote by $\mu_\alpha : K_\alpha^{(G_\alpha)} \subseteq F_\alpha^{(G_\alpha)}$ the embedding which is a direct sum of G_α-many copies of the embedding $K_\alpha \subseteq F_\alpha$. Denote by $\varphi_\alpha : K_\alpha^{(G_\alpha)} \to P_\alpha$ the universal map (that is, the A-homomorphism such that for each $g \in G_\alpha$, the restriction of φ_α to the gth component of $K_\alpha^{(G_\alpha)}$ equals g). Consider the pushout of μ_α and φ_α:

$$\begin{tikzcd}
0 \arrow[r] & K_\alpha^{(G_\alpha)} \arrow[r, "\subseteq"] \arrow[d, "\varphi_\alpha"'] & F_\alpha^{(G_\alpha)} \arrow[r] \arrow[d, "\psi_\alpha"'] & S_\alpha^{(G_\alpha)} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & P_\alpha \arrow[r, "\subseteq"] & P_{\alpha+1} \arrow[r] & S_\alpha^{(G_\alpha)} \arrow[r] & 0
\end{tikzcd}$$

For each limit ordinal $\alpha < \kappa$, we let $P_\alpha = \bigcup_{\beta<\alpha} P_\beta$. Finally, we define $P = \bigcup_{\alpha<\kappa} P_\alpha$. Then there is an exact sequence $0 \to A \to P \to Q \to 0$ where $Q = \bigcup_{\alpha<\kappa} Q_\alpha$, $Q_0 = 0$, $Q_\alpha = \bigcup_{\beta<\alpha} Q_\beta$ for α a limit ordinal $< \kappa$, and $Q_{\alpha+1}/Q_\alpha \cong S_\alpha^{(G_\alpha)}$. Let $T = P \oplus Q$. Then

T is $\leqslant \kappa$-generated. We will prove that $\{T\}^{\perp} = \mathrm{Gen}(T) = \mathcal{S}^{\perp}$—then T is tilting by [7], and satisfies the claim.

First, we prove that $P \in \mathcal{S}^{\perp}$. It suffices to show that for each $\alpha < \kappa$ and each $f \in \mathrm{Hom}_A(K_\alpha, P)$ there is $h \in \mathrm{Hom}_A(F_\alpha, P)$ whose restriction to K_α is f. Since K_α is finitely generated, the image of f is contained in some P_β ($\alpha \leqslant \beta < \kappa$) such that $\mathcal{E}_\beta = \mathcal{E}_\alpha$. We have $f = \sum_{\gamma<\lambda} k_\gamma g_\gamma$ where $G_\beta = \{g_\gamma \mid \gamma < \lambda\}$, and $k_\gamma \in k$ is zero for almost all $\gamma < \lambda$. The restriction of $\psi_\beta : F_\beta^{(G_\beta)} \to P_{\beta+1}$ to $K_\beta^{(G_\beta)}$ is the universal map φ_β. So for each $\gamma < \lambda$, we can use the restriction of μ_β to the g_γth component to obtain $h_\gamma \in \mathrm{Hom}_A(F_\beta, P_{\beta+1})$ such that g_γ is the restriction of h_γ to K_β. Then f is the restriction of $h = \sum_{\gamma<\lambda} k_\gamma h_\gamma$ to K_β. Since $\mathcal{E}_\beta = \mathcal{E}_\alpha$, we infer that $\mathrm{Ext}^1_A(S_\alpha, P) = 0$.

Since T belongs to the torsion class $\mathcal{S}^{\perp}$ we see that $\mathrm{Gen}(T) \subseteq \mathcal{S}^{\perp}$. Since T is an extension of A by $Q^{(2)}$, we also have $\mathcal{S}^{\perp} \subseteq \{T\}^{\perp}$.

Finally, let $M \in \{T\}^{\perp}$. Take an epimorphism $\pi : A^{(\delta)} \to M$, and consider the pushout of π and of the embedding $A^{(\delta)} \subseteq P^{(\delta)}$:

$$\begin{tikzcd}
0 \arrow[r] & A^{(\delta)} \arrow[r, "\subseteq"] \arrow[d, "\pi"'] & P^{(\delta)} \arrow[r] \arrow[d] & Q^{(\delta)} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r, "\subseteq"] \arrow[d] & G \arrow[r] \arrow[d] & Q^{(\delta)} \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Since $M \in \{Q^{(\delta)}\}^{\perp}$, M is a direct summand in G. But G is a homomorphic image of $P^{(\delta)}$, so $M \in \mathrm{Gen}(T)$. $\square$

Given two tilting classes $\mathcal{S}_1^{\perp}$ and $\mathcal{S}_2^{\perp}$ (where $\mathcal{S}_1$ and $\mathcal{S}_2$ are subsets of ind-A), the intersection $\mathcal{S}_1^{\perp} \cap \mathcal{S}_2^{\perp} = (\mathcal{S}_1 \cup \mathcal{S}_2)^{\perp}$ is again a tilting class.

However, even if $\mathcal{S}_1^{\perp} = \{T_1\}^{\perp}$ and $\mathcal{S}_2^{\perp} = \{T_2\}^{\perp}$ where T_1 and T_2 are finitely generated tilting modules, there need not exist any finitely generated tilting module T such that $\{T\}^{\perp} = \mathcal{S}_1^{\perp} \cap \mathcal{S}_2^{\perp}$.

We will first give a criterion for intersection of finitely generated (partial) tilting modules to be of the form $\{T\}^{\perp}$ for a finitely generated tilting module T.

If (X_i) is a countable sequence of finitely generated partial tilting modules, we call this sequence Ext-*ordered* provided that $\mathrm{Ext}^1_A(X_i, X_j) = 0$ for $i \geqslant j$. We will show that for a set $\mathcal{S}$ of finitely generated partial tilting modules which admits an Ext-ordering there always exists a finitely generated tilting module T such that $\{T\}^{\perp} = \mathcal{S}^{\perp}$.

Proposition 2.3. *Let A be a connected Artin algebra. Let $\mathcal{S}$ be a countable set of finitely generated partial tilting modules which admits an* Ext-*ordering. Then there is a finitely generated tilting module T such that $\{T\}^{\perp} = \mathcal{S}^{\perp}$.*

Proof. We may assume that $\mathcal{S}$ already is Ext-ordered. Hence $\mathcal{S} = (X_i \mid i < \sigma)$ where $\sigma \leqslant \omega$, and $\mathrm{Ext}^1_A(X_i, X_j) = 0$ whenever $j \leqslant i < \sigma$.

By induction on $i < \sigma$, we will construct finitely generated partial tilting modules B_i such that $\{B_i\}^\perp = \{X_0, \ldots, X_i\}^\perp$, B_i is a direct summand in B_j, for $i \leqslant j$, and B_i has a filtration

$$0 = Y_{-1} \subseteq Y_0 \subseteq \cdots \subseteq Y_i = B_i$$

such that $Y_j / Y_{j-1} \in \operatorname{add} X_j$.

For $i = 0$, we take $B_0 = X_0$. If B_i is already defined for some $i + 1 \leqslant \sigma$, consider the universal exact sequence in $\operatorname{Ext}^1_A(B_i, X_{i+1})$:

$$0 \to X_{i+1} \to U_i \to B_i^n \to 0,$$

which means that the induced map $\operatorname{Hom}_A(B_i, B_i^n) \to \operatorname{Ext}^1_A(B_i, X_{i+1})$ is surjective, and put $B_{i+1} = B_i \oplus U_i$. By construction has B_{i+1} projective dimension at most 1. The inductive premise yields $\{B_{i+1}\}^\perp = \{X_0, \ldots, X_{i+1}\}^\perp$. The universality then gives $\operatorname{Ext}^1_A(B_i, U_i) = 0$, hence $\operatorname{Ext}^1_A(B_i, B_{i+1}) = 0$. Since $\operatorname{Ext}^1_A(X_{i+1}, B_{i+1}) = 0$, we see that B_{i+1} is partial tilting.

By [6], the number of pairwise non-isomorphic indecomposable summands in any finitely generated partial tilting module is at most the rank of the Grothendieck group of A. So there exists $i_0 < \sigma$ such that $\operatorname{add} B_{i_0} = \operatorname{add} B_j$ for all $i_0 \leqslant j < \sigma$. Then $\{B_{i_0}\}^\perp = \mathcal{S}^\perp$, so again by [6], there is a finitely generated tilting module T such that $\{T\}^\perp = \{B_{i_0}\}^\perp = \mathcal{S}^\perp$. It finally should be mentioned that the partial tilting module B_{i_0} is preinjective (regular, respectively preprojective), provided $\mathcal{S}$ consists of preinjective (regular, respectively preprojective) modules. □

If $\mathcal{S}$ is a (finite) set of finitely generated partial tilting modules which cannot be Ext-ordered, then there may be no finitely generated tilting module T such that $\mathcal{S}^\perp = \{T\}^\perp$:

Example 2.4. (a) Let A be a connected tame hereditary algebra with a tube $\mathcal{T}$ of rank $r > 1$ and let $\mathcal{S} = \{S_i \mid 1 \leqslant i \leqslant r\}$ be the quasi-simple modules in this tube. All S_i are partial tilting modules, but the set $\mathcal{S}$ does not admit an Ext-ordering. $\mathcal{S}^\perp \cap \text{ind-}A$ consists of all indecomposable preinjective modules and all indecomposable modules in the tubes different from $\mathcal{T}$. Therefore there is no finitely generated non-zero Ext-projective module in $\mathcal{S}^\perp$, and $\mathcal{S}^\perp = \{P\}^\perp$ where P is the direct sum of the Prüfer modules belonging to the tube $\mathcal{T}$.

(b) Let A be a connected wild hereditary algebra with at least 3 simple modules. In this case there exists a regular tilting module V, see [22]. Then there exists a natural number t, such that $\mathcal{S} = \{\tau^i V \mid 0 \leqslant i \leqslant t\}$ cannot be Ext-ordered and again in $\mathcal{S}^\perp \cap \text{mod-}A$ there is no indecomposable Ext-projective module. For details see [5, 5.6]. Therefore $\mathcal{S}^\perp = T^\perp$ where T is a tilting module without indecomposable finitely generated direct summands.

3. Preprojective and preinjective modules

In this section, A denotes a connected hereditary algebra of infinite representation type. Since each indecomposable preprojective (preinjective) module is isomorphic to a τ^{-n}-shift (a τ^n-shift) of an indecomposable projective (injective) module for some $n < \omega$, any subset $\mathcal{S} \subseteq \text{ind-}A$ consisting of preprojective or preinjective modules is countable. So Theorem 2.2 applies, but the question remains whether there is a finitely generated tilting module T with $\{T\}^{\perp} = \mathcal{S}^{\perp}$. Our next result provides an answer:

Theorem 3.1. *Let A be a connected hereditary algebra of infinite representation type and let $\mathcal{S}$ be a subset of* ind-A.

(1) *If either $\mathcal{S} \subseteq \mathcal{P}$ or $\mathcal{S} \subseteq \mathcal{I}$, then there is a countably generated tilting module T such that $\{T\}^{\perp} = \mathcal{S}^{\perp}$. T can be taken finitely generated if and only if $\mathcal{S}$ is not an infinite subset of $\mathcal{P}$.*
(2) *If $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ with non-empty sets $\mathcal{S}_1 \subseteq \mathcal{I}$ and $\mathcal{S}_2 \subseteq \mathcal{P}$, then $\mathcal{S}^{\perp} = \{T\}^{\perp}$ for a finitely generated tilting module T.*

Proof of part (1). By Theorem 2.2, there is a countably generated tilting module T with $\{T\}^{\perp} = \mathcal{S}^{\perp}$. We have to show that T can be taken finitely generated if $\mathcal{S}$ is either a finite set of indecomposable preprojective modules or a set of indecomposable preinjective modules. We will show that $\mathcal{S}$ admits an Ext-ordering in these two cases, so Proposition 2.3 applies.

If $\mathcal{S} = \{X_i \mid 1 \leqslant i \leqslant n\}$ is a finite set of indecomposable preprojective modules, then $X_i = \tau^{-a_i} P_{a_i}$, with P_{a_i} indecomposable projective and $a_i \geqslant 0$. Take an ordering such that $a_i \geqslant a_j$ for $i < j$. For $n \geqslant i \geqslant j$ we get $\operatorname{Ext}^1_A(X_i, X_j) \cong \operatorname{D}\operatorname{Hom}_A(\tau^{-a_j-1} P_{a_j}, \tau^{-a_i} P_{a_i}) \cong \operatorname{D}\operatorname{Hom}_A(\tau^{a_i-a_j-1} P_{a_j}, P_{a_i}) = 0$, since $\tau^{a_i-a_j-1} P_{a_j}$ is not projective.

Let $\mathcal{S} = \{I_i \mid i < \sigma\}$ where $\sigma \leqslant \omega$, $I_i = \tau^{\beta_i} Q_{\beta_i}$ for some $\beta_i \geqslant 0$ and Q_{β_i} indecomposable injective for each $i < \sigma$. Order $\mathcal{S}$ by $\beta_{i-1} \leqslant \beta_i$ for all $0 < i < \sigma$.

Then $\operatorname{D}\operatorname{Ext}^1_A(I_i, I_j) = 0$ whenever $j \leqslant i < \sigma$. Indeed, we have $\operatorname{Ext}^1_A(I_i, I_j) \cong \operatorname{Hom}_A(\tau^{\beta_j} Q_{\beta_j}, \tau^{\beta_i+1} Q_{\beta_i}) \cong \operatorname{Hom}_A(Q_{\beta_j}, \tau^{\beta_i+1-\beta_j} Q_{\beta_i})$. The latter group is zero because $\tau^{\beta_i+1-\beta_j} Q_{\beta_i}$ is not injective. $\square$

We postpone the proof of part (2) of Theorem 3.1 till the end of this section, since we will first need the following more detailed discussion of part (1).

In the case when $\mathcal{S}$ is an infinite set of indecomposable preprojective modules, we get

Lemma 3.2. *Let A be a connected hereditary algebra and $\mathcal{S}$ be an infinite subset of $\mathcal{P} \cap$* ind-A. *Then $\mathcal{S}^{\perp} = \mathcal{P}^{\perp}$ is the class of all $\mathcal{P}^{\infty}$-torsion modules.*

Proof. Since $\mathcal{S}$ is infinite, there exists $P \in \mathcal{S}$ for which the set $P_{\mathcal{S}} = \{S \in \mathcal{S} \mid S \cong \tau^{-n} P$ for some $n < \omega\}$ is infinite. By [17, Lemma 6.2(b)], for all $P, Q \in \mathcal{P}$ there is $m < \omega$ such that for each $m \leqslant n < \omega$, Q embeds into $(\tau^{-n} P)^{k_n}$ for some $k_n < \omega$. By assumption on $P_{\mathcal{S}}$, we infer that $P_{\mathcal{S}}^{\perp} \subseteq \{Q\}^{\perp}$, hence $P_{\mathcal{S}}^{\perp} = \mathcal{S}^{\perp} = \mathcal{P}^{\perp}$. The final assertion follows from $\mathcal{P}^{\perp} = \operatorname{Ker}\operatorname{Hom}_A(-, \mathcal{P})$, cf. [17,21]. $\square$

In the setting of Lemma 3.2, Lukas [17, 6.1] gave an explicit construction of a number of equivalent countably generated tilting modules generating $\mathcal{P}^{\perp}$ (compare this with the general construction in Theorem 2.2):

Let $P \neq 0$ be any preprojective module. By induction, we define a chain of preprojective modules A_n ($n < \omega$) as follows: $A_0 = A$; given A_n, we use [21, Lemma 2.5] to construct a short exact sequence $0 \to A_n \subseteq A_{n+1} \to P_n \to 0$ where $A_{n+1}, P_n \in \mathcal{P}$ and $\mathrm{Hom}_A(A_{n+1}, \tau^{-n} P) = 0$. Put $A_P = \bigcup_{n<\omega} A_n$, $B_P = A_P/A$, and $T_P = A_P \oplus B_P$.

Proposition 3.3. *Let A be a connected hereditary algebra of infinite representation type. Let P be a non-zero preprojective module. Then T_P is a tilting module generating the class of all $\mathcal{P}^\infty$-torsion modules.*

Proof. Since $P_n \in \mathcal{P}$ for all $n < \omega$, we have $\mathcal{P}^\perp \subseteq \{T_P\}^\perp$. As in the final paragraph of the proof of Theorem 2.2, we get $\{T_P\}^\perp \subseteq \mathrm{Gen}(T_P)$.

Take any preprojective module Q. We will show that $\mathrm{Ext}^1_A(Q, A_P) = 0$ (then we have $\mathrm{Ext}^1_A(Q, T_P) = 0$, so $\mathrm{Gen}(T_P) \subseteq \{Q\}^\perp$, and $\mathrm{Gen}(T_P) \subseteq \mathcal{P}^\perp$). By the Auslander–Reiten formula, we have to prove that $\mathrm{Hom}_A(A_P, \tau Q) = 0$. If this is not the case, there is $m < \omega$ such that for each $m \leqslant p < \omega$, $\mathrm{Hom}_A(A_p, \tau Q) \neq 0$. However, by [17, Lemma 6.2(b)], there is $n \geqslant m$ such that τQ is cogenerated by $\tau^{-q} P$ for all $q \geqslant n$. In particular, $\mathrm{Hom}_A(A_{n+1}, \tau^{-n} P) \neq 0$, in contradiction with the construction of A_P.

This proves that $\mathcal{P}^\perp = \{T_P\}^\perp = \mathrm{Gen}(T_P)$, so T_P is tilting by [7]. □

Note that there exists no finitely generated tilting module T generating the class of all $\mathcal{P}^\infty$-torsion modules. Indeed, $\mathcal{T} = \mathcal{P}^\perp \cap \text{mod-}A$ is the union of the classes of all regular and all preinjective modules, so $\mathcal{T}$ is not generated by a single module.

If $\mathcal{S}$ is an infinite subset of $\mathcal{I} \cap \text{ind-}A$, we know that $\mathcal{S}^\perp = \{T\}^\perp$ for a finitely generated tilting module T. We will now consider the structure of T in more detail:

Proposition 3.4. *Let A be a connected hereditary algebra of infinite representation type. Let $\mathcal{S}$ be an infinite subset of $\mathcal{I} \cap \text{ind-}A$. Let T be a finitely generated tilting module with $\mathcal{S}^\perp = \{T\}^\perp$. Then T has no indecomposable preprojective direct summands, and $\mathcal{S}^\perp \cap \text{ind-}A$ is finite.*

Moreover, if A is wild then T is preinjective and $\mathcal{S}^\perp \cap \text{mod-}A \subseteq \mathcal{I}$.

Proof. We show that no non-zero preprojective module is in $\mathcal{S}^\perp$. Suppose Y is an indecomposable preprojective module in $\mathcal{S}^\perp$. Since there are only finitely many indecomposable non-sincere preprojective modules, there is some natural number r with $\tau^{-m} Y$ sincere, for $m > r$. By the Auslander–Reiten formula $\mathrm{Hom}_A(\tau^- Y, \mathcal{S}) = 0$. Since $\mathcal{S}$ is an infinite set of indecomposable preinjective modules, it contains a module $\tau^a Q$ with Q indecomposable injective and $a > r$. Then $\mathrm{Hom}_A(\tau^- Y, \tau^a Q) = 0$, hence $\mathrm{Hom}_A(\tau^{-1-a} Y, Q) = 0$, which is a contradiction. Consequently T has no indecomposable preprojective direct summand.

In the same way, again using Proposition 1.2 one shows that no indecomposable regular module is in $\{T\}^\perp$ provided that A is wild. Hence T is preinjective in the latter case and only finitely many modules in ind-A are generated by T, all of them are preinjective.

If A is tame, T may have a regular direct summand. But still it generates only finitely many modules in ind-A by [11]. □

Proof of part (2) of Theorem 3.1. By the proofs of part (1) and of Proposition 2.3, there exists a finitely generated preinjective partial tilting module T_1 with $\mathcal{S}_1^\perp = \{T_1\}^\perp$.

Similarly, if $\mathcal{S}_2$ is finite, there exists a preprojective partial tilting module T_2 with $\mathcal{S}_2^{\perp} = T_2{}^{\perp}$. Since $\mathrm{Ext}_A^1(T_2, T_1) = 0$, Proposition 2.3 applies: there exists a finitely generated tilting module T such that $\{T\}^{\perp} = \{T_1\}^{\perp} \cap \{T_2\}^{\perp} = \mathcal{S}^{\perp}$.

If $\mathcal{S}_2$ is infinite, then $\mathcal{S}_2^{\perp} = \mathcal{P}^{\perp}$ by Lemma 3.2, hence $\mathcal{S}_1^{\perp} \cap \mathcal{S}_2^{\perp} \cap \text{mod-}A = \mathcal{S}_1^{\perp} \cap (\mathcal{R} \vee \mathcal{I})$. We will use a variation of the following fact, which should be well known and is very easy to prove: Let A be an Artin algebra and X a finitely generated partial tilting module. Let L_1 and L_2 be tilting modules in mod-A such that $\{X\}^{\perp} \subset \{L_i\}^{\perp}$, for $i = 1, 2$. Consider the universal exact sequences $0 \to L_i \to M_i \to X^{s_i} \to 0$ in $\mathrm{Ext}_A^1(X, L_i)$. Then $T_i = X \oplus M_i$ are tilting A-modules with $\operatorname{add} T_1 = \operatorname{add} T_2$.

We consider $\mathcal{S}_1^{\perp} \cap \mathcal{P} = \mathcal{Y}$. By Proposition 1.2(A), $\mathcal{Y} \cap \text{ind-}A$ is a finite set (otherwise, the preprojective component of $\Gamma(A)$ contains an infinite set of non-sincere modules), possibly it is empty. Since it is a finite subset of $\mathcal{P}$, for $m \gg 0$ and all $Y \in \mathcal{Y}$, we get $\mathrm{Ext}_A^1(\tau^{-m} A, Y) \neq 0$. The preprojective tilting module $T_2 = \tau^{-m} A$ generates all regular and preinjective modules, and the indecomposable preprojective modules of the form $\tau^{-s} P$, for $s \geqslant m$ and P indecomposable projective. Hence $\mathcal{S}_2^{\perp} \cap \mathcal{S}_1^{\perp} \cap \text{mod-}A = \{T_2\}^{\perp} \cap \mathcal{S}_1^{\perp} \cap \text{mod-}A = \{T_2\}^{\perp} \cap \{T_1\}^{\perp} \cap \text{mod-}A$. Again, we can apply Proposition 2.3 since $\mathrm{Ext}_A^1(T_2, T_1) = 0$ and Theorem 2.1. $\square$

4. Regular modules

In this section, except for Proposition 4.1, A denotes a connected wild hereditary algebra. We continue by considering the case when $\mathcal{S} \subseteq \text{ind-}A$ consists of regular modules.

If $\mathcal{S} = \{X_i \mid i \in I\}$ where $\mathrm{Ext}_A^1(X_i, X_j) = 0$ for all $i, j \in I$ then $X = \bigoplus_{i \in I} X_i$ is a partial tilting module, so [6] gives that $\mathcal{S}^{\perp} = \{T\}^{\perp}$ for a finitely generated tilting module T. However, if the partial tilting modules X_i are not Ext-orthogonal then $\mathcal{S}^{\perp}$ need not be of the form $\{T\}^{\perp}$ for any finitely generated tilting module – see Example 2.4.

Next, we consider the case when $\mathcal{S} = \{X_i \mid i \in I\}$ where $\mathrm{Ext}_A^1(X_i, X_i) \neq 0$ for all $i \in I$:

Proposition 4.1. *Let A be a connected hereditary algebra of infinite representation type. Let $\mathcal{S}$ be any set consisting of indecomposable regular modules such that $\mathrm{Ext}_A^1(X, X) \neq 0$ for all $X \in \mathcal{S}$. Then there is no finitely generated tilting module T with $\mathcal{S}^{\perp} = \{T\}^{\perp}$.*

Proof. Assume there is such T. Then $T = \bigoplus_{i \leqslant m} T_i \oplus P$ where P is projective, each T_i ($i \leqslant m$) is indecomposable non-projective partial tilting. By [10, Corollary 4.2] we can additionally assume that $\mathrm{Hom}_A(T_i, T_j) = 0$ whenever $i < j \leqslant m$.

Let $Y = \bigoplus_{X \in \mathcal{S}} X$. Since $\mathrm{Ext}_A^1(T_0, \tau T_0) \neq 0$, then $\mathrm{Ext}_A^1(Y, \tau T_0) \neq 0$. The Auslander–Reiten formula gives $\mathrm{Hom}_A(T_0, Y) \neq 0$, see Lemma 1.1. Let $f : T_0 \to Y$ be a non-zero map.

Since T is a finitely generated tilting module over an Artin algebra, T is product-complete, and hence a cotilting module in the sense of [3], that is, ${}^{\perp}\{T\}$ coincides with the class of all modules cogenerated by T. Since $\mathrm{Ext}_A^1(Y, T) = 0$, there is a monomorphism $Y \hookrightarrow T^{\kappa}$ for a cardinal κ. Since $\mathrm{Hom}_A(Y, P) = 0$, there is a monomorphism $g : Y \hookrightarrow (\bigoplus_{i \leqslant m} T_i)^{\kappa}$.

The composition gf is a non-zero homomorphism from T_0 to $(\bigoplus_{i \leqslant m} T_i)^\kappa$. Since $\mathrm{Hom}_A(T_0, T_i) = 0$ for $i > 0$, there is a projection $p_0 : (\bigoplus_{i \leqslant m} T_i)^\kappa \to T_0$ such that $h_0 = p_0 gf$ is a non-zero endomorphism of T_0. By [10, Lemma 4.1], the endomorphism ring of T_0 is a field, so h_0 is an automorphism. Then f is a split monomorphism, so T_0 is isomorphic to a direct summand in X, a contradiction. $\square$

Another case when $\mathcal{S}^\perp \neq \{T\}^\perp$ for any finitely generated tilting module T is the one when $\mathcal{S}$ contains copies of infinitely many shifts in the τ-direction of a fixed regular module. Following [17], we will show that $\mathcal{S}^\perp$ is then the class of all divisible modules $\mathcal{D} = \mathcal{R}^\perp$:

Lemma 4.2. *Let A be a connected wild hereditary algebra. Let $\mathcal{S} \subseteq$ ind-A be such that there exists a non-zero regular module R for which the set $\mathcal{O} = \{S \in \mathcal{S} \mid S \cong \tau^n R$ for some $n < \omega\}$ is infinite. Then $\mathcal{S}^\perp = \mathcal{D}$.*

Proof. Let $M \in \mathcal{O}^\perp$. It suffices to prove that M is divisible. To this purpose, take any $N \in \mathcal{R}$. By [17, Theorem 2.3], there is $m < \omega$ such that for each $m \leqslant n < \omega$, the module τN embeds into $\tau^{n+1} R$. So $\mathrm{Ker}\,\mathrm{Hom}_A(-, \tau^{n+1} R) \subseteq \mathrm{Ker}\,\mathrm{Hom}_A(-, \tau N)$, and by the Auslander formula, $\{\tau^n R\}^\perp \subseteq \{N\}^\perp$. By assumption on $\mathcal{O}$, there is $p \geqslant m$ such that $\tau^p R$ is isomorphic to a module in $\mathcal{O}$, so $M \in \{N\}^\perp$. This proves that $M \in \mathcal{R}^\perp = \mathcal{D}$. $\square$

Of course, there is no finitely generated tilting module T with $\{T\}^\perp = \mathcal{D}$. Again following [17], we will now construct rather different, but equivalent, countably generated tilting modules generating the tilting class $\mathcal{D}$. (This contrasts with the non-equivalence of the Ringel tilting modules T_S defined in the introduction in the tame case for the sets S of tubes. The construction should again be compared with the one in Theorem 2.2.)

Let R be any non-zero regular module. By induction, we define a chain M_n $(n < \omega)$ of finitely generated modules as follows: $M_0 = A$; given M_n, we use [17, Lemma 2.5] to construct a short exact sequence $0 \to M_n \subseteq M_{n+1} \to R_n \to 0$ where $R_n \cong \tau^r R^l$ for some $r, l < \omega$ and $\mathrm{Hom}_R(M_{n+1}, \tau^n R) = 0$. Put $M_R = \bigcup_{n<\omega} M_n$, $N_R = M_R/A$, and $T_R = M_R \oplus N_R$.

Proposition 4.3. *Let A be a connected wild hereditary algebra and R be a non-zero regular module. Then T_R is a tilting module generating the class of all divisible modules.*

Proof. That $\mathcal{R}^\perp \subseteq \{T_R\}^\perp \subseteq \mathrm{Gen}(T_R)$ follows similarly as in the proof of Proposition 3.3. It remains to show that $\mathrm{Ext}_A^1(Q, M_R) = 0$, or equivalently, $\mathrm{Hom}_A(M_R, \tau Q) = 0$, for all $Q \in \mathcal{R}$. If this is not the case, there is $m < \omega$ such that for each $m \leqslant p < \omega$, $\mathrm{Hom}_A(M_p, \tau Q) \neq 0$. However, by Proposition 1.2(B), there is $n \geqslant m$ such that τQ embeds into $\tau^q R$ for all $q \geqslant n$. In particular, $\mathrm{Hom}_A(M_{n+1}, \tau^n R) \neq 0$, in contradiction with the construction of M_R. $\square$

In particular, if $\mathcal{S} = \{X\}$ where X is an indecomposable regular module, then there exists a finitely generated tilting module T such that $\mathcal{S}^\perp = \{T\}^\perp$ if and only if X is partial tilting. Anyway, in contrast with Lemma 4.2, the tilting class $\mathcal{S}^\perp$ is much bigger than $\mathcal{D}$:

Lemma 4.4. *Let A be a connected wild hereditary algebra. Let R and S be non-zero regular modules. Then there exists $m < \omega$ such that $\{\tau^p S\}^\perp \subsetneq \{R\}^\perp$ for all $m \leqslant p < \omega$. Moreover, there is $n < \omega$ such that $\{R\}^\perp \cap \mathcal{R} \subsetneq \{\tau^{-q} S\}^\perp \cap \mathcal{R}$ for all $n \leqslant q < \omega$.*

Proof. As in the proof of Lemma 4.2, [17, Theorem 2.3] yields an $m < \omega$ such that for each $m \leqslant m' < \omega$, $\{\tau^{m'} S\}^\perp \subseteq \{R\}^\perp$. Moreover, Proposition 1.2(B) gives an $r < \omega$ such that $\mathrm{Hom}_A(\tau^{r'} S, \tau R) = 0$ for all $r \leqslant r' < \omega$, hence $\tau^{r'} S \in \{R\}^\perp$. In particular, taking $m, r - 1 \leqslant p < \omega$, we have $\{\tau^p S\}^\perp \subsetneq \{R\}^\perp$, since $\tau^{p+1} S \notin \{\tau^p S\}^\perp$. Swapping the roles of R and S, and using the fact that τ^{-1} is an equivalence on $\mathcal{R}$, we get the second part of the claim. □

Proposition 4.5. *Let A be a connected wild hereditary algebra and R a non-zero regular module.*

(1) $\{R\}^\perp \cap (\mathcal{R} \vee \mathcal{I}) = (\mathcal{P} \cup \{R\})^\perp \cap \text{mod-}A$. *If $\tau^i R$ is sincere for all $i \geqslant 1$, then* $\{R\}^\perp \cap (\mathcal{R} \vee \mathcal{I}) = \{R\}^\perp \cap \text{mod-}A$.

(2) *There exists $m < \omega$ such that there is a strictly increasing chain*

$$\{R\}^\perp \cap (\mathcal{R} \vee \mathcal{I}) \subsetneq \{\tau^{-m} R\}^\perp \cap (\mathcal{R} \vee \mathcal{I}) \subsetneq \{\tau^{-2m} R\}^\perp \cap (\mathcal{R} \vee \mathcal{I}) \subsetneq \cdots. \tag{$*$}$$

The chain $(*)$ extends to a strictly increasing chain of tilting classes whose supremum is $\mathcal{P}^\infty$:

$$\big(\{R\} \cup \mathcal{P}\big)^\perp \subsetneq \big(\{\tau^{-m} R\} \cup \mathcal{P}\big)^\perp \subsetneq \big(\{\tau^{-2m} R\} \cup \mathcal{P}\big)^\perp \subsetneq \cdots. \tag{$**$}$$

If $\tau^i R$ is sincere for all integers i, then $(**)$ coincides with the chain

$$\{R\}^\perp \subsetneq \{\tau^{-m} R\}^\perp \subsetneq \{\tau^{-2m} R\}^\perp \subsetneq \cdots. \tag{$***$}$$

Otherwise the supremum of the set $\{\{\tau^{-rm} R\}^\perp \mid r \geqslant 0\}$ coincides with $\{T\}^\perp$ for a finitely generated preprojective tilting module T.

(3) *There is a strictly decreasing chain of tilting classes whose intersection is $\mathcal{D}$:*

$$\cdots \subsetneq \{\tau^{p_n} R\}^\perp \subsetneq \cdots \subsetneq \{\tau^{p_1} R\}^\perp \subsetneq \{R\}^\perp$$

for some $0 < p_1 < \cdots < p_n < \cdots$. If $\tau^i R$ is sincere for all integers i, we can take $p_r = r \cdot m$ $(r \geqslant 1)$.

Proof. (1) The first statement follows from $\mathcal{P}^\perp \cap \text{mod-}A = \mathcal{R} \vee \mathcal{I}$. If $\tau^i R$ is sincere for all $i > 0$, then $\{R\}^\perp \cap \mathcal{P} = 0$, and vice versa. Indeed, for an indecomposable preprojective module $Q = \tau^{-s} P$, where P is indecomposable projective, $\mathrm{Ext}^1_A(R, Q) \cong \mathrm{Hom}_A(P, \tau^{s+1} R)$.

(2) By Proposition 1.2, there is $t < \omega$ such that for all $u \geqslant t$, and each regular module S, $\mathrm{Hom}_A(\tau^{-u} S, \tau R) = 0$ implies $\mathrm{Hom}_A(S, \tau R) = 0$. Using the Auslander–Reiten formula,

and the fact that τ is an equivalence on $\mathcal{R}$, we get $\{\tau^u R\}^\perp \cap \mathcal{R} \subseteq \{R\}^\perp \cap \mathcal{R}$. Again, using the equivalence, we infer that $\{\tau^{-k} R\}^\perp \cap \mathcal{R} \subseteq \{\tau^{-l} R\}^\perp \cap \mathcal{R}$ whenever $l \geqslant k + t$.

By Proposition 1.2(B), there is $m \geqslant t$ such that $\mathrm{Hom}_A(R, \tau^{-m} R) = 0$. Then $\mathrm{Ext}^1_A(\tau^{-(n+1)m} R, \tau^{-nm+1} R) = 0$, but clearly $\mathrm{Ext}^1_A(\tau^{-nm} R, \tau^{-nm+1} R) \neq 0$. This proves that $(\tau^{-nm} R)^\perp \cap \mathcal{R} \subsetneq (\tau^{-(n+1)m} R)^\perp \cap \mathcal{R}$. Since $\mathcal{I} \subseteq S^\perp$ for any regular module S, the chain $(*)$ is strictly increasing.

Denote by $\mathcal{T}$ the supremum of the set, $\{\{\tau^{-rm} R\}^\perp \mid r \geqslant 0\}$, of tilting classes in Mod-A. If X is any regular module, then Proposition 1.2(B) yields an $r \geqslant 0$ such that $\mathrm{Hom}_A(X, \tau^{-rm+1} R) = 0$. This implies that $\mathcal{R} \subseteq \mathcal{T}$ and consequently $\mathcal{R} \cup \mathcal{I} \subseteq \mathcal{T}$. Hence $\mathcal{P}^\perp \subseteq \mathcal{T}$ by Theorem 2.1 which shows that the strictly increasing chain $(**)$ has supremum $\mathcal{P}^\perp = \mathcal{P}^\infty$. If R is τ-sincere (that is, all $\tau^i R$ are sincere), then $(**)$ coincides with $(***)$ by (1).

If R is not τ-sincere then $\mathcal{T}_0 = \mathcal{T} \cap \text{mod-}A$ contains all regular and preinjective, and additionally some indecomposable preprojective, modules. So $\mathcal{T}_0$ is generated by a preprojective tilting module by [5], and so is $\mathcal{T}$.

(3) This follows by Lemma 4.2, the first part of Lemma 4.4, and by (the proof of) part (2). $\square$

If R is any non-zero regular module and $\mathcal{S}$ any infinite subset of $\{\tau^r R \mid r \geqslant 0\}$, then from Lemma 4.2 we infer that $\mathcal{S}^\perp$ is the class of all divisible modules. The following example deals with infinite subsets of $\{\tau^{-r} R \mid r \geqslant 0\}$ and shows a rather different behavior:

Example 4.6. Let A be a connected wild hereditary algebra, let R be an indecomposable regular module and $\mathcal{S} = \{\tau^{-p_i} R \mid 0 = p_0 < p_1 < \cdots\}$ an infinite set.

(a) If $\mathrm{Ext}^1_A(R, R) \neq 0$, then by Theorem 2.2 and Proposition 4.1, $\mathcal{S}^\perp$ is the tilting class of a countably generated tilting module, but not of a finitely generated one.

(b) If A has at least three pairwise non-isomorphic simple modules, then there exist infinitely many regular components in the Auslander–Reiten quiver $\Gamma(A)$ containing quasi-simple modules without self-extensions. Let R be one of these modules. It is a brick, by [10]. Therefore $\mathrm{Hom}_A(\tau^i R, \tau^j R) = 0$ for all integers with $i > j$, see Proposition 1.2(B). Choosing a strictly increasing sequence (p_i) of natural numbers with $p_{i+1} - p_i \geqslant 2$, the sequence $(\tau^{-p_i} R)$ is Ext-ordered. Hence $\{\tau^{-p_i} R \mid i < \omega\}^\perp$ is the tilting class of a finitely generated tilting module T, by Proposition 2.3.

5. Irredundant modules, and a reduction procedure

In this section, except for Lemmas 5.2 and 5.3, A denotes a hereditary algebra. We continue by considering the question of when $\{R\}^\perp = \{S\}^\perp$ where R and S are indecomposable regular modules. In general, this can occur even if $R \ncong S$:

Example 5.1. Let A be a hereditary algebra of infinite representation type and M be a (regular) brick such that $\mathrm{Ext}^1_A(M, M) \neq 0$. Then there exists a chain of indecomposable regular modules N_n $(1 \leqslant n < \omega)$ such that $N_1 = M$ and $N_{i+1}/N_i \cong M$ for all $1 \leqslant i < n$, see [19]. By Lemma 1.3, the tilting classes $\{M\}^\perp$ and $\{N_n\}^\perp$ coincide for all $1 \leqslant n < \omega$, but the modules $N_1 = M, N_2, N_3, \ldots$ are pairwise non-isomorphic.

In order to compare the tilting torsion classes $R^\perp$ and $S^\perp$ for $R, S \in \text{mod-}A$, we will use the following more general lemma:

Lemma 5.2. *Let A be a right hereditary ring, $M, N \in \text{Mod-}A$ be such that M is noetherian. The following are equivalent*:

(1) $\{N\}^\perp \subseteq \{M\}^\perp$.
(2) *There exists $k < \omega$ and a chain $M_0 \subseteq \cdots \subseteq M_k = M$ of submodules of M such that M_0 is projective, and M_{i+1}/M_i is isomorphic to a submodule of N for each $i < k$.*

If A is a hereditary Artin algebra, then these conditions are also equivalent to

(3) $\{N\}^\perp \cap \text{mod-}A \subseteq \{M\}^\perp \cap \text{mod-}A$.

Proof. (1) implies (2): Consider a short exact sequence $0 \to G \to F \to M \to 0$ with F finitely generated and free. By [9], there is an exact sequence $0 \to G \to X \to Y \to 0$ such that $X \in \{N\}^\perp$ and there is a continuous chain $(Y_\alpha \mid \alpha \leqslant \kappa)$ consisting of submodules of Y such that $Y_0 = 0$, $Y_\kappa = Y$, and $Y_{\alpha+1}/Y_\alpha \cong N$ for $\alpha < \kappa$. Consider the pushout of the monomorphisms $G \to F$ and $G \to X$:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & G \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & M \arrow[r] & 0 \\
 & Y \arrow[r, equal] \arrow[d] & Y \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

By (1), $M \in {}^\perp(\{N\}^\perp)$, so the second row splits, and without loss of generality, M is a direct summand in H. The second column yields a continuous chain $(H_\alpha \mid \alpha \leqslant \kappa)$ consisting of submodules of H such that $H_0 \cong F$, $H_\kappa = H$, and $H_{\alpha+1}/H_\alpha \cong N$ for $\alpha < \kappa$. Let $M_\alpha = M \cap H_\alpha$. Then $M_0 \subseteq H_0$ is projective, and $M_{\alpha+1}/M_\alpha \cong ((M \cap H_{\alpha+1}) + H_\alpha)/H_\alpha \subseteq H_{\alpha+1}/H_\alpha \cong N$. Since M is noetherian, there are only finitely many different members of the chain $(M_\alpha \mid \alpha \leqslant \kappa)$, and the claim follows.

(2) implies (1): For each $i < k$, take $S_i \subseteq N$ such that $S_i \cong M_{i+1}/M_i$. Then $\{N\}^\perp \subseteq \{S_i\}^\perp$, so $\{N\}^\perp \subseteq \bigcap_{i<k} \{S_i\}^\perp$. On the other hand, by induction on $j \leqslant k$, we have $\bigcap_{i<j} \{S_i\}^\perp \subseteq \{M_j\}^\perp$, so $\bigcap_{i<k} \{S_i\}^\perp \subseteq \{M\}^\perp$.

It remains to prove that (3) implies (1) in case R is a hereditary Artin algebra. Let $\mathcal{S} = {}^{\perp}(\{N\}^{\perp}) \cap \text{mod-}R$. Clearly, $\{N\}^{\perp} \subseteq \mathcal{S}^{\perp}$.

On the other hand, by Eilenberg's trick, there is an exact sequence $0 \to G \to F \to N \to 0$ where F and G are free of infinite rank. Let $(g_i \mid i \in I)$ be a free basis of G and, for each finite subset $J \subseteq I$, let $G_J = \bigoplus_{j\in J} g_j R$. Then G is the directed union of the direct system $(G_J \mid J \subseteq I, J$ finite). The induced direct system of finitely presented modules $(N_J \mid J \subseteq I, J$ finite) satisfies $N \cong H \oplus \varinjlim_J N_J$ where H is free. It is easy to see that $N_J \in \mathcal{S}$ for each finite subset $J \subseteq I$.

Since each module $P \in \mathcal{S}^{\perp} \cap \text{mod-}R$ is pure-injective, we have $\operatorname{Ext}^1_R(N, P) \cong \varprojlim_J \operatorname{Ext}^1_R(N_J, P) = 0$. It follows that $\{N\}^{\perp} \cap \text{mod-}A = \mathcal{S}^{\perp} \cap \text{mod-}A$.

By Theorem 2.1, (3) implies that $\mathcal{S}^{\perp} \subseteq \{M\}^{\perp}$, so $\{N\}^{\perp} \subseteq \{M\}^{\perp}$. $\square$

Clearly, $\{M\}^{\perp} = \{N\}^{\perp}$ whenever M and N are any projective modules. We will consider a case when $\{M\}^{\perp} = \{N\}^{\perp}$ implies $M \cong N$ for indecomposable modules:

Let M be a non-zero noetherian module over a ring A. Then M is *irredundant* if $\{M\}^{\perp} \neq \bigcap_{N\in\mathcal{F}} \{N\}^{\perp}$ for each finite set, $\mathcal{F}$, which consists of submodules of M, but does not contain M. Clearly, any irredundant module is non-projective and indecomposable.

Lemma 5.3. *Let A be a right hereditary ring and M, N be irredundant modules of finite length with $\{M\}^{\perp} = \{N\}^{\perp}$. Then $M \cong N$.*

Proof. By Lemma 5.2, there exists $k < \omega$ and a chain $M_0 \subseteq \cdots \subseteq M_k = M$ of submodules of M such that M_0 is projective, and M_{i+1}/M_i is isomorphic to a submodule S_i of N for each $i < k$. If $S_i \neq N$ for all $i < k$, then $\{N\}^{\perp} \subsetneq \bigcap_{i<k} \{S_i\}^{\perp} \subseteq \{M\}^{\perp}$, a contradiction. So there exists $i < k$ with $S_i \cong N$. Similarly, M is isomorphic to a subfactor of N. Since M and N are of finite length, we have $M \cong N$. $\square$

For Artin algebras, the irredundant modules coincide with the non-projective bricks:

Lemma 5.4. *Let A be a hereditary Artin algebra and M be a non-zero finitely generated module. Then M is irredundant if and only if M is a non-projective brick.*

Proof. Assume M is irredundant. Denote by B the A-endomorphism ring of M. Then the Jacobson radical of B is nilpotent, say of degree n. If $n > 1$, there is $0 \neq f \in B$ such that $f^2 = 0$. Let K and I denote the kernel and image of f, respectively. Then $I \subseteq K \subsetneq M$, in particular, $\{M\}^{\perp} \subseteq \{K\}^{\perp} \subseteq \{I\}^{\perp}$. The exact sequence $0 \to K \to M \to I \to 0$ yields $\{K\}^{\perp} = \{K\}^{\perp} \cap \{I\}^{\perp} \subseteq \{M\}^{\perp}$. So $\{M\}^{\perp} = \{K\}^{\perp}$, and M is not irredundant. This proves that $n = 1$, that is, the local ring B is a skew-field.

The reverse implication holds for an arbitrary Artin algebra A: Let M be a non-projective brick. First, we prove that $\tau M \in \{U\}^{\perp}$ for any proper submodule $U \subsetneq M$. Since M is a brick and $U \neq M$, we have $\operatorname{Hom}_A(M, U) = 0$. By the Auslander–Reiten formula, there is an epimorphism $\operatorname{Hom}_A(M, U) \to \operatorname{D}\operatorname{Ext}^1_A(U, \tau M)$. Hence $\operatorname{Ext}^1_A(U, \tau M) = 0$.

Finally, let $\mathcal{F}$ be a finite set of proper submodules of M. Then $\tau M \in \bigcap_{U\in\mathcal{F}} \{U\}^{\perp}$ but $\operatorname{Ext}^1_A(M, \tau M) \neq 0$. This proves that M is irredundant. $\square$

Theorem 5.5. *Let A be a hereditary Artin algebra. Let M and N be non-projective bricks such that $\{M\}^{\perp} = \{N\}^{\perp}$. Then $M \cong N$.*

Proof. By Lemmas 5.3 and 5.4. □

If X is a finitely generated non-projective A module, then it is irredundant if and only if it is a brick. If it is not a brick, then there exists a finite set of proper submodules $\mathcal{F}$ such that $\{X\}^{\perp} = \mathcal{F}^{\perp}$. More can be shown:

Proposition 5.6. *Let A be a hereditary Artin algebra and X a module of finite length. Then there exist bricks $S_1, \dots, S_t$ with $\mathrm{Hom}_A(S_i, S_j) = 0$ for $i \neq j$ such that $Y = \bigoplus_{1 \leqslant i \leqslant t} S_i$ is a submodule of X with $\{Y\}^{\perp} = \{X\}^{\perp}$.*

Proof. The proof is by induction on the composition length $c(X)$ of X. If $c(X) = 1$, we take $Y = X$. Assume the statement holds for all modules of composition length smaller than n and take X with $c(X) = n$.

If $\mathrm{rad}\,\mathrm{End}(X) = (0)$ then X is a direct sum $X = \bigoplus_{1 \leqslant i \leqslant t} S_i^{n_i}$ of pairwise orthogonal bricks S_i, and we choose $Y = \bigoplus_{1 \leqslant i \leqslant t} S_i$. If $\mathrm{End}(X)$ is not semi-simple, we choose $0 \neq f$ in the radical of $\mathrm{End}(X)$ with $f^2 = 0$. As in the proof of Lemma 5.4, we get $\{X\}^{\perp} = \{\mathrm{Ker}\, f\}^{\perp}$. Since $\mathrm{Ker}\, f$ is a submodule of X with smaller composition length, induction applies for $\mathrm{Ker}\, f$. □

The module $Y = \bigoplus_{1 \leqslant i \leqslant t} S_i$ can possibly be reduced further: If S_j is a direct summand of Y such that $\{\bigoplus_{i \neq j} S_i\}^{\perp} \subseteq \{S_j\}^{\perp}$, the direct summand S_j of Y can be omitted.

Example 5.7. If A is tame hereditary, $\mathcal{T}$ is a tube of rank $r \geqslant 1$ and Y is an indecomposable module in $\mathcal{T}$, then there exists a chain of irreducible monomorphisms $S = S(1) \to S(2) \to \cdots \to S(m) = Y$, where S is a simple regular module. For $m < r$, the module Y has no self-extensions, therefore $\{Y\}^{\perp}$ is a tilting torsion class defined by a finitely generated tilting module. If $m \geqslant r$, then it is well known that $\{Y\}^{\perp} = \{S(r)\}^{\perp}$.

If A is a connected wild hereditary algebra, some weaker analog still holds true: Let $\mathcal{C}$ be a regular component in the Auslander–Reiten quiver $\Gamma(A)$ of mod-A and let X be quasi-simple in this component. (For technical statements used in the sequel, we refer to the survey [16].)

(a) Take $m \geqslant 1$ such that $\mathrm{Hom}_A(X, \tau^t X)$ contains a monomorphism for all $t \geqslant m$, see Proposition 1.2(B). Take a monomorphism $f : X \to Y$, where $Y = \tau^{t-1} X$, for $t - 1 \geqslant m$, and consider the chain of irreducible monomorphisms

$$Y = Y(1) \to Y(2) \to \cdots \to Y(t-1) \to Y(t)$$

which we consider as embeddings. Denote by $\pi : Y(t) \to X$ the cokernel of the embedding $Y(t-1) \to Y(t)$. The map $g = f\pi : Y(t) \to Y(t)$ has kernel $Y(t-1)$ and image in Y. Hence by the proof of Proposition 5.6, we get $\{Y(t)\}^{\perp} = \{Y(t-1)\}^{\perp}$.

(b) Let X additionally be an elementary module, which means that the kernels of non-zero homomorphisms from X to any regular module R are preprojective. Then X is a brick and all modules $\tau^i X$ in the τ-orbit of X are elementary, too. Since in the τ-orbit of X there are at most finitely many non-sincere modules, after some τ-shift we may assume that $\tau^i X$ is sincere, for all $i \geqslant 0$. There exists a number $1 \leqslant r \leqslant n-1$, where n denotes the number of simple A-modules, such that $X(r)$ is a brick with self-extensions, the modules $X(i)$ with $i < r$ have no self-extensions and all the modules $X(i)$ with $i > r$ have non-trivial endomorphism rings [12,15]. For $i > r$, denote by $\pi : X(i) \to Z = \tau^{-i+1} X$ the cokernel of the irreducible embedding $e : X(i-1) \to X(i)$. There exists a non-zero homomorphism $f_0 : Z \to \tau^r Z$ and this homomorphism can be lifted to a non-zero homomorphism $f : Z \to X(i-r)$. Indeed, since $\tau^r Z = \tau^{r-i-1} X$, there exists a chain of irreducible epimorphisms $\pi' : X(i-r) \to \tau^r Z$, therefore [20, 4.6*] applies. Let $g = f\pi$. Since e is an irreducible map and $X(i-1) \subset \operatorname{Ker} g \neq X(i)$, we get $\operatorname{Ker} g = X(i-1) \oplus Y$. Moreover we have $Y \cong \pi(\operatorname{Ker} g) = \operatorname{Ker} f$. Clearly $g^2 = 0$, hence $\{X(i)\}^{\perp} = \{(X(i-1) \oplus \operatorname{Ker} f)\}^{\perp}$. Since Z is elementary, $\operatorname{Ker} f$ is preprojective. Since $\tau^j X(i-1)$ is sincere for all $j \geqslant 0$, we get $\{X(i-1)\}^{\perp} \subset \{\operatorname{Ker} f\}^{\perp}$. Therefore $\{X(i)\}^{\perp} = \{X(i-1)\}^{\perp}$, for all $i > r$.

(c) If A is connected wild hereditary with two simple modules, all indecomposable regular modules are sincere and have self-extensions. In this case, we get for each elementary module X and any natural number $i \geqslant 1$, similarly to the case of the (tame) Kronecker algebra, $\{X\}^{\perp} = \{X(i)\}^{\perp}$.

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